PMM U.S.S.R.,Vol.45.pp.575-580
Copyright Pergamon Press Ltd.1982.Printed in U.K.

0021-8928/82/5 0575 $7.50/0

UDC 531.36

APPLICATIONS OF THE METHODS OF AVERAGING AND SUCCESSIVE APPROXIMATIONS FOR STUDYING NONLINEAR OSCILLATIONS*

L.D. AKULENKO

The question is examined of the existence of a solution of the Cauchy problem for a system of ordinary differential equations, standard in the sense of Bogoliubov /1—3/, describing a wide class of nonlinear processes of oscillations and rotations. Constructive sufficient conditions for the existence and uniqueness of this solution on asymptotically large time intervals are established by the method of successive approximations /4,5/. Smoothness properties of a nonstationary process with respect to the problem parameters (initial data) are investigated.

1. Statement of the problem and initial assumptions. We examine a standard system of ordinary differential equations on an asymptotically large time interval, with prescribed initial conditions /1—3/

$$x^{\cdot} = \varepsilon X(t, x),\ x(t_0) = x^\circ,\ t \in [t_0, T],\ T = \Theta\varepsilon^{-1} \tag{1.1}$$

Here x, X are vectors of arbitrary dimension $n \geqslant 1$, t_0, x°, Θ are prescribed constants, $\varepsilon > 0$ is a small numerical parameter, $\varepsilon \in [0, \varepsilon_0]$. The following requirements are assumed of the vector function X.

1) It is defined for all $t \geqslant t_0$ and $x \in D_x$, where $D_x \subset E^n$ is an open connected set, and is 2π-periodic and piecewise-continuous in t,

2) Function X has a continuous derivative with respect to $x \in D_x$, satisfying a Lipschitz condition in the domain indicated.

3) For all $\varepsilon \in (0, \varepsilon_0]$, $x^\circ \in D_{x^\circ} \subset D_x$ there exists a unique solution of Cauchy problem (1.1) (this will be proved later)

$$x = x(t, t_0, x^\circ, \varepsilon),\quad t \in [t_0, T] \tag{1.2}$$

belonging to the open domain D_x.

4) The Cauchy problem for the averaged system (1.1)

$$d\xi / d\tau = X_0(\xi),\ \xi(0) = x^0,\ \varepsilon t = \tau \in [\tau_0, \Theta] \tag{1.3}$$

where X_0 is the average of function X with respect to t on the period 2π, τ is slow time, admits of the general solution

$$\xi = \xi(\tau - \tau_0, x^\circ),\ \xi \in D_x,\ x^\circ \in D_{x^\circ} \tag{1.4}$$

in the domain indicated. This solution is assumed known.

There are a large number of studies (see /1—3/ and the bibliography in /3/) establishing the proximity property, in particular, ε-proximity, between the unknown solution (1.2) of the original Cauchy problem (1.1) and the solution (1.4) of the averaged system (1.3) under the same initial conditions. The averaged system is autonomous and permits the exclusion of parameter ε; its solution is usually studied or can be constructed in a manner essentially more simpler than the original complete system. The determination of the unknown general solution (1.2) to a higher degree of accuracy with respect to ε on the asymptotically large time interval $t \sim \varepsilon^{-1}$ is difficult since the constructions of the averaging method, connected with a change of variables, leads to the solving of a perturbed system of partial differential equations. The latter circumstance calls for a corresponding high degree of smoothness of function X with respect to $x \in D_x$. In the paper we examine the existence and uniqueness of the general solution (1.2) of system (1.1) under the fulfillment of assumptions 1), 2) and 4). A constructive algorithm is developed for the construction of this solution and its properties are studied relative to changes in the problem parameters.

*Prikl.Matem.Mekhan.,45,No.5,771-777,1981

2. Simplification of the original Cauchy problem. In system (1.1) we make a number of changes of the unknown variable x. Analogously to /1— 3/ we introduce the averaging method's error δ

$$\delta = x - \xi_*, \ \xi_*(t, \varepsilon) = \xi(\tau) + \varepsilon u(t, \varepsilon) \tag{2.1}$$
$$u(t, \varepsilon) = \int_{t_0}^{t} [X(s, \xi(\sigma)) - X_0(\xi(\sigma))]\, ds, \quad \sigma = \varepsilon s$$

Here function u is uniformly bounded in t, ε; the dependency of u, ξ on t_0, x° in (2.1) is not indicated for brevity. For the unknown δ we obtain the Cauchy problem

$$\delta^{\cdot} = \varepsilon X'(t, \xi)\,\delta + \varepsilon\Phi(t, \delta, \varepsilon), \ \delta(t_0) = 0 \tag{2.2}$$
$$\Phi = [X(t, \xi_*) - X(t, \xi)] + [X(t, \xi_* + \delta) - X(t, \xi_*) - X'(t, \xi), \delta] \tag{2.3}$$

Here Φ is a known function of t, δ, ε. As follows from (2.3), the estimate

$$|\Phi| \leqslant c_\Phi(\varepsilon + \varepsilon|\delta| + \delta^2), \ c_\Phi = \text{const} \tag{2.4}$$
$$|\Phi(t, \delta_1, \varepsilon) - \Phi(t, \delta_2, \varepsilon)| \leqslant \lambda_\Phi |\delta_1 - \delta_2| (|\delta_1| + |\delta_2| + \varepsilon), \ \lambda_\Phi = \text{const}$$

is valid for Φ in domain $(\xi_* + \delta) \in D_x$. We make an almost-identity transformation /6/ of vector δ

$$\delta = (I + \varepsilon U)z, \quad U(t, \varepsilon) \equiv \int_{t_0}^{t} [X'(s, \xi(\sigma)) - X_0'(\xi(\sigma))]\, ds \tag{2.5}$$

Here I is the unit matrix, U is a uniformly-bounded matrix-valued function analogous to u from (2.1); the prime denotes a derivative with respect to vector x. Differentiating substitution (2.5) relative to system (2.2), we obtain the Cauchy problem

$$z^{\cdot} = \varepsilon X_0'(\xi)\, z + \varepsilon F(t, z, \varepsilon), \ z(t_0) = 0 \tag{2.6}$$
$$F \equiv (I + \varepsilon U)^{-1} \{[X(t, \xi_*) - X(t, \xi)] - \varepsilon U X_0'(\xi)\, z + \varepsilon X'(t, \xi)\, Uz + [X(t, \xi_* + (I + \varepsilon U)z) - X(t, \xi_*) - X'(t, \xi_* - \varepsilon u)\, z]\} \tag{2.7}$$

for the unknown z. Here F is a known function of t, z, ε; according to (2.7) an estimate of type (2.4) is valid for it, on the basis of assumption 2) of Sect.1, when $[\xi_* + (I + \varepsilon U)z] \in D_x$, and a Lipschitz condition is fulfilled

$$|F| \leqslant c_F(\varepsilon + \varepsilon|z| + z^2), \ c_F = \text{const} \tag{2.8}$$
$$|F(t, z_1, \varepsilon) - F(t, z_2, \varepsilon)| \leqslant \lambda_F |z_1 - z_2| (|z_1| + |z_2| + \varepsilon),$$
$$\lambda_F = \text{const}$$

Using estimate (2.8) a perturbation by a quantity of the order of ε^2 can be made. Indeed, function F can be presented in the form

$$F(t, z, \varepsilon) = \varepsilon F_\varepsilon(t, \varepsilon) + \varepsilon F_{\varepsilon z}(t, z, \varepsilon)\, z + F_{z^2}(t, z, \varepsilon)\, z^2 \tag{2.9}$$

where $F_\varepsilon, F_{\varepsilon z}, F_{z^2}$ are uniformly bounded functions. Observe that when $F \equiv 0$, (2.6) is a system in variations for (1.3). Allowing for the properties listed, in system (2.6) we make a change of the unknown variable z

$$z = \varepsilon Z(r + \varphi), \quad Z(\tau) \equiv \partial\xi/\partial x^\circ, \quad \varphi(t, \varepsilon) \equiv \int_{t_0}^{t} Z^{-1}(\sigma)\, F(s, 0, \varepsilon)\, ds \tag{2.10}$$

Here Z is the fundamental matrix for the variational system mentioned, φ is a known uniformly-bounded function, and r is an unknown variable. On the strength of (2.8) and (2.9) the integral in (2.10) is uniformly bounded. As a result, we obtain the Cauchy problem

$$r^{\cdot} = \varepsilon R(t, r, \varepsilon), \ r(t_0) = 0 \tag{2.11}$$

for the unknown vector r. Function R is uniformly bounded and satisfies a Lipschitz condition

$$|R| \leqslant \varepsilon c_R, \ c_R = \text{const} \tag{2.12}$$
$$|R(t, r_1, \varepsilon) - R(t, r_2, \varepsilon)| \leqslant \varepsilon\lambda_R |r_1 - r_2|, \ \lambda_R = \text{const}$$

To construct the solution of Cauchy problem (2.11) on an asymptotically large time interval $t \in [t_0, T]$, $T = \Theta\varepsilon^{-1}$, we apply the recurrence procedure of the method of successive approximations (Picard's method) with respect to powers of ε /4,5/.

3. Construction of the exact solution. The unknown function r is constructed by use of the recurrence scheme of Picard's method /7/

$$r_{j+1}(t,\varepsilon)=\varepsilon\int_{t_0}^{t} R(s,r_j(s,\varepsilon)\varepsilon)\,ds,\quad r_0\equiv 0,\quad j=0,1,\ldots \tag{3.1}$$

Note that the variable r depends on the parameters t_0 and x° as well; however, this dependence is not indicated for brevity. The following assertion is valid.

Theorem. For a sufficiently small $\varepsilon>0$ the successive approximations (3.1) converge uniformly to a unique solution of Cauchy problem (2.11) on the asymptotically large interval $t\in[t_0,\ T]$.

Let us prove that any j_*-th approximation is bounded by the quantity εM, where $M>0$ is some constant defined below. Indeed, we represent r_{j_*} as the sum

$$r_{j*}=r_0+(r_1-r_0)+(r_2-r_1)+\ldots+(r_{j_*}-r_{j_*-1}) \tag{3.2}$$

We make use of (2.12) and of the estimation

$$|r_1|\leqslant\varepsilon N,\quad N=\max_t|Z^{-1}[F(t,\varepsilon Z\varphi,\varepsilon)-F(t,0,\varepsilon)]|\,\Theta\varepsilon^{-2}$$

Majorizing sum (3.2), for $t\in[t_0,\ \Theta\varepsilon^{-1}]$ with the values $\varepsilon\leqslant\varepsilon_0<\lambda_R/\Theta$ we obtain

$$|r_{j_*}|\leqslant\sum_{j=1}^{j_*}|r_j-r_{j-1}|\leqslant\sum_{j=1}^{\infty}|r_j-r_{j-1}|\leqslant\varepsilon N\Big[1+\sum_{j=1}^{\infty}(\varepsilon\lambda_R\Theta)^j\Big]=\frac{\varepsilon N}{1-\varepsilon\lambda_R\Theta}\leqslant\varepsilon M \tag{3.3}$$

From (3.1) and (3.3) it follows as well that the successive approximations converge uniformly when $\varepsilon<1/\lambda_R\Theta$ to a continuous differentiable function r_*

$$\lim_{j\to\infty}r_j=r_*(t,\varepsilon),\quad \lim_{j\to\infty}r_j^{\cdot}=r_*^{\cdot}(t,\varepsilon) \tag{3.4}$$

Let us show that the limit function $r_*(t,\varepsilon)$ from (3.4) is a solution of the Cauchy problem (2.11). This assertion follows from the fact that the solution $\rho(t,\varepsilon)$ of system

$$\rho^{\cdot}=\varepsilon R\,(t,\ r_*\,(t,\ \varepsilon),\ \varepsilon),\ \rho\,(t_0)=0,\ t\in[t_0,\ T]$$

coincides with $r_*(t,\ \varepsilon)$.

Finally, the solution $r_*(t,\varepsilon)$ constructed is unique, which can be proved by contradiction. Let ρ_1 and ρ_2 be any two solutions of Cauchy problem (2.11); then from the identities

$$\rho_i(t,\varepsilon)\equiv\varepsilon\int_{t_0}^{t}R(s,\rho_i(s,\varepsilon),\varepsilon)\,ds,\quad i=1,2$$

and from (2.12) follows the inequality

$$\max_t|\rho_1-\rho_2|\leqslant\varepsilon\lambda_R\Theta\max_t|\rho_1-\rho_2|,\quad \varepsilon\lambda_R\Theta<1$$

which is possible only when $\rho_1\equiv\rho_2$ for all $t\in[t_0,\ T]$. The theorem has been proved. Thus, substituting the function $r_*(t,\ \varepsilon)$ constructed into (2.10), next, the known $z(t,\ \varepsilon)$ into expression (2.5) for δ, and, finally, the known function $\delta(t,\ \varepsilon)$ into (2.1), we obtain the required exact solution of Cauchy problem (1.1) in the form (η is a uniformly bounded function)

$$x\,(t,\ t_0,\ x^\circ,\ \varepsilon)=\xi\,(\tau-\tau_0,\ x^\circ)+\varepsilon\eta\,(t,\ t_0,\ x^\circ,\ \varepsilon) \tag{3.5}$$
$$\eta\equiv u+(I+\varepsilon U)\,Z\,(r_*+\varphi)$$

4. Notes. 1°. Using the replacements

$$x=\xi^*+\delta,\ \xi^*(t,\ \tau,\ \varepsilon)=\xi\,(\tau)+\varepsilon v\,(t,\ \xi,\ (\tau)) \tag{4.1}$$
$$v=v\,(t,\xi)=\int_{t_0}^{t}[X\,(s,\ \xi)-X_0\,(\xi)]\,ds$$

where v and ξ^* are 2π-periodic functions of the explicitly occurring argument t, the original equation system (1.1) is reduced to form (2.2) with another function $\Phi=\Phi^*(t,\tau,\delta,\varepsilon)$. This function is 2π-periodic in t and in the domain being examined satisfies in ε and δ the uniform estimate (2.4) since

$$\Phi \equiv X(t, \xi^* + \delta) - X(t, \xi) - X'(t, \xi)\delta - \varepsilon^2 V' X_0(\xi) \tag{4.2}$$

Here the repeated differentiability of X with respect to x is assumed. With the aid of a linear almost-identity transformation of the unknown δ we obtain a system of type (2.6)

$$\begin{gathered} y^{\cdot} = \varepsilon X_0'(\xi) y + \varepsilon G(t, \tau, y, \varepsilon), \quad y(t_0) = 0 \\ \delta = [I + \varepsilon V(t, \xi)]\, y, \; V(t, \xi) \equiv \partial v / \partial \xi, \; \xi = \xi(\tau) \\ G \equiv (I + \varepsilon V)^{-1} [\Phi + \varepsilon X'(t, \xi) V y - \varepsilon V' X_0 y - \varepsilon V X_0' y] \end{gathered} \tag{4.3}$$

Here function G is 2π-periodic in the explicitly occurring argument t. Obviously, an estimate of type (2.8) in ε and y is valid for G and G can be represented in form (2.9). A scheme of successive approximations in powers of ε, analogous to that developed in section 3, is applicable to system (4.3). As the first approximation of the solution, vanishing when $\varepsilon = 0$ for all $t \in [t_0, T]$, we take the function

$$y_0 = \varepsilon \varphi(t, \varepsilon) \equiv \varepsilon Z(\tau) \int_{t_0}^{t} Z^{-1}(\sigma)\, G(s, \sigma, 0, \varepsilon)\, ds, \; \sigma = \varepsilon s \tag{4.4}$$

where φ is a uniformly bounded function on the interval indicated. The recurrence scheme of Picard's method enables us to construct by quadratures the required solution $y(t, \varepsilon)$ to any preassigned degree of accuracy in ε

$$y_{j+1}(t, \varepsilon) = \varepsilon Z(\tau) \int_{t_0}^{t} Z^{-1}(\sigma)\, G(s, \sigma, y_j(s, \varepsilon), \varepsilon)\, ds, \quad j = 0, 1, \ldots \tag{4.5}$$

The successive approximations (4.4), (4.5) converge uniformly for a sufficiently small $\varepsilon > 0$ to the unique solution of Cauchy problem (4.3). Thus, again we have constructed the desired exact solution of form (4.1) of the Cauchy problem (1.1). The approach suggested above can be more suitable for the analytical construction of functions v and V, as well as Φ and G, for example, as a Fourier series in t. However, as follows from (4.2), it requires a somewhat higher smoothness property of the function X relative to $x \in D_x$.

2°. Using the approach in /7/ we can establish the continuous differentiability of solution (3.5) with respect to the initial data t_0, x°. At first we establish the uniform continuity of the limit function $r(t, \varepsilon, \mu)$, and, together with it, of the solution x of the original system (1.1) with respect to some parameter $\mu, \mu \in D_\mu$, if function X depends continuously on μ. This assertion follows from the uniform convergence of sequence (3.4), obtained by Picard's method (3.1). On the strength of the fact of uniform continuity we next prove the possibility of passing to the limit as $\Delta \to 0$ in the matrix differential equation

$$\frac{d}{dt}\left(\frac{x_\Delta - x}{\Delta}\right) = \varepsilon A(t, \Delta) \frac{x_\Delta - x}{\Delta} \tag{4.6}$$

Here x_Δ is the solution of system (1.1) under the initial condition $x_\Delta(t_0) = x^\circ + \Delta$. The matrix A, depending continuously on parameter Δ, is obtained on the basis of the finite increment theorem. As a result of the limit transitions in (4.6) we obtain the linear matrix equation

$$P^{\cdot} = \varepsilon X'(t, x(t, \varepsilon))\, P, \; P(t_0) = I \tag{4.7}$$

for the matrix P of partial derivatives of solutions x with respect to the initial data x° ($P = \partial x / \partial x^\circ$). Here I is the unit matrix. For each i-th column p_i of matrix P we obtain a variational system /4/ with appropriate initial conditions

$$p_i^{\cdot} = \varepsilon X'(t, x)\, p_i, \; p_{ii}(t_0) = 1, \; p_{ik}(t_0) = 0 \; (k \neq i) \tag{4.8}$$

An equation analogous to (4.8) is obtained for the vector of the derivative of the solution with respect to t_0; the initial value equals /7/ $-\varepsilon X(t_0, x^\circ)$.

We seek the solution of Cauchy problem (4.7) as

$$P = (I + \varepsilon U_*)\, W, \quad U_*(t, \varepsilon) = \int_{t_0}^{t} [X'(s, x(s, \varepsilon)) - X_0'(\xi(\sigma))]\, ds \tag{4.9}$$

Here W is an unknown matrix and U_* is a matrix uniformly bounded for $t \in [t_0, T]$. Differentiating substitution (4.9) relative to system (4.3), we obtain

$$(I+\varepsilon U_*)W^{\cdot}=\varepsilon X_0'W+\varepsilon^2 X'U_*W,\ W(t_0)=I \tag{4.10}$$

We represent the required matrix W as a series with unknown matrix coefficients $C_j\ (j=1,2,\ldots)$

$$W=Z(I+\varepsilon C_1+\varepsilon^2 C_2+\ldots),\ C_j(t_0)=0 \tag{4.11}$$

Substituting this expression into (4.10), we obtain recurrence relations for the uniformly bounded coefficients $C_j(t,\varepsilon)$

$$C_{j+1}=\int_{t_0}^{t} Z^{-1}[\varepsilon(X'U_*-X_0')ZC_j-U_*ZC_j^{\cdot}]\,ds \tag{4.12}$$

$$C_1=\varepsilon\int_{t_0}^{t} Z^{-1}[X'(s,\xi,(\sigma))U_*(s,\varepsilon)-X_0'(\xi(\sigma))]Z\,ds,\quad j=1,2,\ldots$$

For a sufficiently small $\varepsilon>0$ the series (4.11) converges to the unique solution of system (4.10) for all $t\in[t_0,T]$. By the same token we have defined, in accord with (4.9), a matrix P characterizing the sensitivity of solution $x(t,t_0,x^\circ,\varepsilon)$ of (3.5) with respect to changes in the initial data.

In analogous fashion we can determine "a sensitivity matrix" K for the solution $x(t,t_0,x^\circ,\varepsilon,\mu)$ in the general case of a system of form

$$x^{\cdot}=\varepsilon X(t,x,\mu),\ x(t_0)=x^\circ,\ t_0=t_0(\mu),\ x^\circ=x^\circ(\mu),\ \mu\in D_\mu$$

with respect to the parameter vector μ of arbitrary dimension m, which can include the initial data. To do this we can solve, by the means proposed above, the Cauchy problem for the matrix equation with initial conditions obtained by differentiation of functions $x^\circ(\mu)$, $t_0(\mu)$ with respect to μ

$$K^{\cdot}=\varepsilon X'(t,x,\mu)K+\varepsilon\partial X(t,x,\mu)/\partial\mu$$

$$K=\frac{Dx}{D\mu},\quad K(t_0)=\frac{\partial x^\circ}{\partial\mu}-\varepsilon X(t_0,x^\circ,\mu)\frac{\partial t_0}{\partial\mu}$$

In particular, if parameters x° and t_0 are fixed and are independent of μ, we obtain a linear inhomogeneous matrix equation with zero initial conditions.

3°. Let us now consider the Cauchy problem for a standard system with rotating phase on an asymptotically large interval $t\in[t_0,T]$ (see /1,3,6,8,9/)

$$a^{\cdot}=\varepsilon A(a,\psi),\ \ \psi^{\cdot}=\omega(a)+\varepsilon\Psi(a,\psi),\ a(t_0)=a^\circ,\ \psi(t_0)=\psi^\circ \tag{4.13}$$

Here a is the n-vector of slow variables, ψ is the scalar phase, A and Ψ are 2π-periodic functions of the fast variable ψ, and the frequency $\omega(a)\geqslant\omega_0>0$. The right-hand sides of Eqs.(4.13) are taken to be sufficiently smooth with respect to $a\in D_a,\psi\in[\psi^\circ,\infty)$. We observe that system (4.13) describes a wide class of perturbed essentially-nonlinear rotary-oscillatory processes. By the usual device /8/, the division of $a^{\cdot}$ by $\psi^{\cdot}$, we obtain a standard system of form (1.1)

$$\frac{da}{d\psi}=\varepsilon\frac{A(a,\psi)}{\omega(a)+\varepsilon\Psi(a,\psi)},\quad a(\psi^\circ)=a^\circ \tag{4.14}$$

We accept the fulfillment of the assumptions in section 1 for system (4.14). Then on an asymptotically large interval of variation of the fast variable $\psi,\psi\in[\psi^\circ,\psi_T],\psi_T\sim\varepsilon^{-1}$, we obtain, by using the approach in sections 2 and 3, a solution, smooth with respect to ψ° and $a^\circ\in D_{a^\circ}\subseteq D_a$ of a form analogous to (3.5)

$$a=a(\psi,\psi^\circ,a^\circ,\varepsilon)=\alpha(\theta-\theta^\circ,a^\circ)+\varepsilon\gamma(\psi,\psi^\circ,a^\circ,\varepsilon)$$

$$\theta=\varepsilon\psi,\ \theta\in[\theta^\circ,\theta_T],\ \theta_T\sim 1$$

Here α is a solution of the first-approximation system averaged with respect to ψ

$$d\alpha/d\theta=A_0(\alpha)/\omega(\alpha),\ \ \alpha(\theta^\circ)=a^\circ$$

and γ is a function uniformly bounded for all $\psi\in[\psi^\circ,\psi_T]$.

Substituting the expression for a into the equation for ψ in (4.13), we obtain an equation with separable variables, uniquely connecting ψ and t

$$t-t_0=\int_{\psi^\circ}^{\psi}\frac{d\varphi}{\omega(\alpha)+\varepsilon B(\varphi)},\quad \varepsilon B\equiv\omega(\alpha+\varepsilon\gamma)-\omega(\alpha)+\varepsilon\Psi \tag{4.15}$$

For brevity we do not indicate the dependence of the bounded function B on the other arguments. From relation (4.15) the required function $\psi(t)$ and, together with it, $a(t)$ can be found by successive approximations based on the first-approximation solution. The quantity $T \sim \varepsilon^{-1}$ is determined from (4.15) for the value $\psi = \psi_T$.

Thus, the successive approximations method (Picard's method) suggested above proves to be a rather effective means of analyzing nonstationary oscillatory processes on a large time interval. The results obtained by its use have an independent significance and are particularly important when solving the boundary-value problems of Pontriagin's maximum principle; a number of optimal control problems for nonlinear oscillatory systems lead to the investigation of such boundary-value problems when asymptotic methods /9/ are used.

REFERENCES

1. BOGOLIUBOV N.N. and MITROPOL'SKII Iu.A., Asymptotic Methods in the Theory of Nonlinear Oscillations. Gordon & Breach, New York, 1964.
2. BESJES J.G., On the asymptotic methods for nonlinear differential equations. J. Mécanique, Vol.8, No.3, 1969.
3. MITROPOL'SKII Iu.A., The Averaging Method in Nonlinear Mechanics. Kiev, NAUKOVA DUMKA, 1971.
4. MALKIN I.G., Certain Problems in the Theory of Nonlinear Oscillations. Moscow, Gostekhizdat, 1956.
5. GREBENIKOV E.A. and RIABOV Iu.A., Constructive Analysis Methods for Nonlinear Systems. Moscow, NAUKA, 1979.
6. BOGOLIUBOV N.N., MITROPOL'SKII Iu.A. and SAMOILENKO A.M., The Method of Accelerated Convergence in Nonlinear Mechanics. Kiev. NAUKOVA DUMKA, 1969.
7. STEPANOV V.V., Course on Differential Equations. Moscow, Gostekhizdat, 1953.
8. VOLOSOV V.M. and MORGUNOV B.I., The Averaging Method in the Theory of Nonlinear Oscillatory Systems. Moscow, Izd. Mosk. Gos. Univ., 1971.
9. AKULENKO L.D., Approximate solution of nonlinear problems of optimal control of oscillatory processes using the method of canonical separation of motions. PMM Vol.40, No.6,1976.

Translated by N.H.C.